International Journal of Theoretical Physics, Vol. 33, No. 6, 1994

Representations of Empirical Set Theories

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Received September 23, 1993

Any manual $\mathfrak{M}$ of Boolean locales in the strong sense, namely a subcategory of the opposite category $\mathfrak{BLoc}$ of the category $\mathfrak{Bool}$ of complete Boolean algebras and complete Boolean homomorphisms satisfying not only conditions (3.1)–(3.10) of our previous paper [*International Journal of Theoretical Physics*, **32**, 1293 (1993*b*)], but also conditions (4.1)–(4.4) of that paper, is shown to be representable as the second-class orthomodular manual $\mathfrak{M}_{[\mathcal{Q}]}$ of Boolean locales on an orthomodular poset $\mathcal{Q}$. In this sense the study on manuals of Boolean locales in the strong sense is tantamount to the study on a special class of orthomodular posets, though our viewpoint is radically different from the conventional one in the traditional approach to orthomodular posets. Then the notion of a manual of Hilbert spaces or exactly what is called a manual of Hilbert locales is introduced, over which a variant of the celebrated Gelfand–Naimark–Segal theorem for a manual of Boolean locales in the strong sense is established.

1. INTRODUCTION

Some researchers in the foundations of quantum mechanics have made something of partial structures such as partial Boolean algebras, partial Hilbert spaces, partial semigroups, and so on. (Czelakowski, 1974, 1975, 1978, 1979, 1981; Gudder, 1972, 1986; Kochen and Specker 1965*a,b*; Lock and Hardegree, 1985*a,b*; Mączyński, 1970). We agree completely with them that partiality lies at the core of quantum theory, but not until we put it within the modern framework of category theory does it become truly comprehensible. We believe that but for the modern apparatus of category theory, the study on partial structures would be doomed to remain parochial.

In a previous paper (Nishimura, 1993*b*), sharing the same operational metaphysics with Foulis and Randall (1972; Randall and Foulis, 1973) we introduced the notion of a manual of Boolean algebras or exactly what was

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0020-7748/94/0600-1173$07.00/0

called a manual of Boolean locales, which we believe will supersede partial Boolean algebras and the like. In Section 3 of this paper we will introduce the notion of a manual of Hilbert spaces or exactly what is to be called a manual of Hilbert locales, which we believe incarnate precisely what the advocates of partial Hilbert spaces have tried to express. In Section 4 we present a representation theorem of manuals of Boolean locales over manuals of Hilbert locales after the celebrated Gelfand–Naimark–Segal (GNS) theorem, which has given a canonical method for producing representations of operator algebras over Hilbert spaces, and which dates back to Gelfand and Naimark (1943) and Segal (1947). In Section 2 we discuss representations of manuals of Boolean locales over orthomodular posets.

We hold firmly that the notion of a manual is more fundamental, more extensive, and far and away more pregnant than even its original proponents presumably envisaged. We feel that manuals stand in the same position to empirical mathematics as sheaves stand to constructive or intuitionistic mathematics, though empirical mathematics is just beginning to bloom. For the predominance of sheaves in model theory of constructive mathematics the reader is referred, e.g., to Trooelstra and van Dalen (1988, Chapters 14 and 15 in particular). In subsequent papers Nishimura (n.d.-*a*,*b*) we will discuss manuals of operator algebras and commutative algebras. We expect that it will not be very long before the theory of manuals becomes as indispensable a tool in the arsenal of every working mathematician, ranging from algebraic geometry to analysis, as the theory of sheaves has already.

We assume that the reader is well conversant with our previous paper (Nishimura, 1993*b*). A manual of Boolean locales is said to be a manual of Boolean locales *in the strong sense* if it satisfies conditions (4.1)–(4.4) besides obligatory conditions (3.1)–(3.10) of that paper. Hilbert spaces always imply complex Hilbert spaces.

2. REPRESENTATION OF MANUALS OF BOOLEAN LOCALES OVER ORTHOMODULAR POSETS

The consideration in Example 3.7 of Nishimura (1993*b*) already contained the following.

Proposition 2.1. Let $\mathscr{Q}$ be an arbitrary orthomodular poset. Then the orthomodular poset $\mathscr{Q}(\mathfrak{M}_{[\mathscr{Q}]})$ associated with the second-class orthomodular manual $\mathfrak{M}_{[\mathscr{Q}]}$ of Boolean locales on $\mathscr{Q}$ is naturally isomorphic $\mathscr{Q}$.

Given an orthomodular poset $\mathscr{Q}$, an assignment to each $x \in \mathscr{Q}$ of a nonnegative real number $\omega(x)$ is said to be a *state* if it abides by the following conditions:

(2.1) $\omega(1) = 1$

(2.2) For any family $\{x_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal elements of $\mathcal{Q}$ such that $\sup_{\lambda\in\Lambda} x_\lambda$ exists,

$$\omega(\sup_{\lambda\in\Lambda} x_\lambda) = \sum_{\lambda\in\Lambda} \omega(x_\lambda)$$

Given a manual $\mathfrak{M}$ of Boolean locales, an assignment to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of a nonnegative real number $\omega(\mathbf{X})$ is called a state if it satisfies the following conditions:

(2.3) $\omega(\mathbf{X}) = 1$ for any $\mathfrak{M}$-maximal Boolean locale $\mathbf{X}$ in $\mathfrak{M}$.

(2.4) For any family $\{X_\lambda\}_{\lambda\in\Lambda}$ of mutually $\mathfrak{M}$-orthogonal Boolean locales in $\mathfrak{M}$ such that $\sum_{\lambda\in\Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ exists,

$$\omega\left(\sum_{\lambda\in\Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda\right) = \sum_{\lambda\in\Lambda} \omega(\mathbf{X}_\lambda)$$

It is easy to see by condition (3.9) of Nishimura (1993*b*) that if $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$, then $\omega(\mathbf{X}) = \omega(\mathbf{Y})$.

Note that our notion of a state on an orthomodular set or on a manual of Boolean locales assumes complete additivity from the beginning. It is easy to see the following:

Proposition 2.2. For any orthomodular poset $\mathcal{Q}$, the states on $\mathcal{Q}$ are naturally in bijective correspondence with the states on the second-class manual $\mathfrak{M}_{[\mathcal{Q}]}$ of Boolean locales on $\mathcal{Q}$.

Proof. Let ω be a state on $\mathcal{Q}$. It is easy to see that the assignment $\bar{\omega}$ to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}_{[\mathcal{Q}]}$ of $\bar{\omega}(\mathbf{X}) = \omega(1_{\mathbf{X}})$ with $1_{\mathbf{X}}$ the unit element of the relative complete Boolean subalgebra $\mathscr{P}(\mathbf{X})$ of $\mathcal{Q}$ is a state on $\mathfrak{M}_{[\mathcal{Q}]}$. It is also easy to see that the assignment $\omega \mapsto \bar{\omega}$ gives a bijective correspondence between the states on $\mathcal{Q}$ and the states on $\mathfrak{M}_{[\mathcal{Q}]}$. ■

Recall that an orthomodular poset $\mathcal{Q}$ is called *regular* if any finite family of mutually compatible elements in $\mathcal{Q}$ is contained by a Boolean subalgebra of $\mathcal{Q}$ (Pták and Pulmannová, 1991, Definition 1.3.26 and Proposition 1.3.29). We say that the orthomodular poset $\mathcal{Q}$ is *strongly regular* if any family of mutually compatible elements in $\mathcal{Q}$ is contained by a complete Boolean subalgebra of $\mathcal{Q}$. Since a family of mutually orthogonal elements in $\mathcal{Q}$ is admittedly a family of mutually compatible elements in $\mathcal{Q}$, strong regularity implies orthocompleteness. It is easy to see the following:

Proposition 2.3. For any manual $\mathfrak{M}$ of Boolean locales in the strong sense, the orthomodular poset $\mathcal{Q}(\mathfrak{M})$ associated with $\mathfrak{M}$ is strongly regular.

We can see readily the following result.

Theorem 2.4. Let $\mathscr{Q}$ be an arbitrary orthomodular poset. Then the second-class orthomodular manual $\mathfrak{M}_{[\mathscr{Q}]}$ of Boolean locales on $\mathscr{Q}$ is a manual of Boolean locales in the strong sense iff $\mathscr{Q}$ is strongly regular.

The main representation theorem of this section goes as follows:

Theorem 2.5. Any manual $\mathfrak{M}$ of Boolean locales in the strong sense is naturally equivalent to the second-class orthomodular manual $\mathfrak{M}_{[\mathscr{Q}(\mathfrak{M})]}$ of Boolean locales on the orthomodular poset $\mathscr{Q}(\mathfrak{M})$ associated with $\mathfrak{M}$.

Proof. It is not difficult to see that the assignment to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of the relative complete Boolean subalgebra $\mathbf{B}_{\mathbf{X}}$ depicted in Section 4 of Nishimura (1993*b*) induces an equivalence $F: \mathfrak{M} \to \mathfrak{M}_{[\mathscr{Q}(\mathfrak{M})]}$. ■

Corollary 2.6. Given a manual $\mathfrak{M}$ of Boolean locales in the strong sense, the states on $\mathfrak{M}$ are naturally in bijective correspondence with the states on the orthomodular poset $\mathscr{Q}(\mathfrak{M})$ associated with $\mathfrak{M}$.

Proof. This follows readily from Proposition 2.2 and Theorem 2.5. ■

The gist of the consideration in this section is that strongly regular orthomodular posets and manuals of Boolean locales in the strong sense are the same thing from different viewpoints. Nonetheless we believe that the transition from the conventional vantage point to our categorical one is considerably fertile.

3. MANUALS OF HILBERT LOCALES

We denote by $\mathfrak{Hil}$ the category of Hilbert spaces and contractive linear transformations. That is to say, a linear transformation $T: \mathscr{H} \to \mathscr{K}$ of Hilbert spaces is a morphism of $\mathfrak{Hil}$ iff $\|T(x)\| \leq \|x\|$ for any $x \in \mathscr{H}$. Note that isomorphisms in $\mathfrak{Hil}$ are no other than unitary transformations of Hilbert spaces. A Hilbert space whose dimension is zero is called *trivial.* Recall that a linear transformation $U: \mathscr{H} \to \mathscr{K}$ of Hilbert spaces is called a *partial isometry* if it is isometric on the orthogonal complement of the null space $\{x \in \mathscr{H} \mid Ux = 0\}$. In this case the orthogonal complement of the null space of U is called the *initial space* of U and denoted by $\mathscr{I}(U)$, while the range of U, which is also a closed linear subspace of $\mathscr{K}$, is called the *final space* of U and denoted by $\mathscr{F}(U)$. Note that every partial isometry resides in $\mathfrak{Hil}$.

The opposite category of the category $\mathfrak{Hil}$ is denoted by $\mathfrak{HLoc}$. Its objects are called *Hilbert locales* and denoted by $\mathbf{X}, \mathbf{Y}, \ldots$, while its morphisms are denoted by $\mathbf{f}, \mathbf{g}, \ldots$. Given an object $\mathbf{X}$ of $\mathfrak{HLoc}$, the opposite $\mathbf{X}^{\mathrm{op}}$ of $\mathbf{X}$, which is an object of $\mathfrak{Hil}$, is also denoted by $\mathscr{H}(\mathbf{X})$. Similarly, given a morphism $\mathbf{f}$ of $\mathfrak{HLoc}$, the opposite $\mathbf{f}^{\mathrm{op}}$ of $\mathbf{f}$, which is a morphism of $\mathfrak{Hil}$, is denoted by $\mathscr{H}(\mathbf{f})$. A Hilbert locale $\mathbf{X}$ is called *trivial* if $\mathscr{H}(\mathbf{X})$ is trivial. A morphism $\mathbf{f}: \mathbf{Y} \to \mathbf{X}$ of $\mathfrak{HLoc}$ is called an *embedding* provided that $\mathscr{H}(\mathbf{f})$ is a partial isometry with $\mathscr{F}(\mathscr{H}(\mathbf{f})) = \mathscr{H}(\mathbf{Y})$. Two embeddings $\mathbf{f}: \mathbf{Y} \to \mathbf{X}$ and $\mathbf{f}': \mathbf{Y}' \to \mathbf{X}$ with the same codomain $\mathbf{X}$ are said to be *equivalent* if there exists an isomorphism $\mathbf{g}: \mathbf{Y} \to \mathbf{Y}'$ with $\mathbf{f}' \circ \mathbf{g} = \mathbf{f}$. A morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is called a *surjection* provided that $\mathscr{H}(\mathbf{f})$ is a partial isometry with $\mathscr{I}(\mathscr{H}(\mathbf{f})) = \mathscr{H}(\mathbf{X})$. That is to say, a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a surjection iff $\mathscr{H}(\mathbf{f})$ is an isometric transformation. Two surjections $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ and $\mathbf{f}': \mathbf{X} \to \mathbf{Y}'$ with the same domain $\mathbf{X}$ are said to be *equivalent* if there exists an isomorphism $\mathbf{g}: \mathbf{Y} \to \mathbf{Y}'$ with $\mathbf{g} \circ \mathbf{f} = \mathbf{f}'$. A family

$$\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$$

of morphisms of the $\mathfrak{HLoc}$ with the same codomain is said to be a *partial orthogonal sum diagram* if $\mathbf{f}_\lambda$ is an embedding for each $\lambda \in \Lambda$ and $\mathscr{I}(\mathscr{H}(\mathbf{f}_\lambda))$'s are mutually orthogonal, in which $\mathbf{X}$ is called a *partial orthogonal sum* of $\mathbf{X}_\lambda$'s. The partial orthogonal sum diagram

$$\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$$

is said to be an *orthogonal sum diagram* if $\mathscr{H}(\mathbf{X})$ is the orthogonal sum of $\mathscr{I}(\mathscr{H}(\mathbf{f}_\lambda))$'s, in which $\mathbf{X}$ is called an *orthogonal sum* of $\mathbf{X}_\lambda$'s.

Let $\mathfrak{M}$ be a small subcategory of the category $\mathfrak{HLoc}$. A diagram of $\mathfrak{HLoc}$ is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Hilbert locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$ are said to be $\mathfrak{M}$-*orthogonal*, in notation $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, if there exists a partial orthogonal sum diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$$

of $\mathfrak{HLoc}$ lying in $\mathfrak{M}$. A trivial Hilbert locale in $\mathfrak{M}$ is said to be $\mathfrak{M}$-*trivial* if it is an initial object in $\mathfrak{M}$. A Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$ is said to be $\mathfrak{M}$-*maximal* if for any Hilbert locale $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ implies that $\mathbf{Y}$ is $\mathfrak{M}$-trivial. Hilbert locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$ are said to be $\mathfrak{M}$-equivalent, in notation $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$, provided that for any Hilbert locale $\mathbf{Z}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Obviously $\mathfrak{M}$-equivalence is an equivalence relation among the Hilbert locales in $\mathfrak{M}$. We denote by $[\mathbf{X}]_{\mathfrak{M}}$ the equivalence class of $\mathbf{X}$ with respect to $\mathfrak{M}$-equivalence. An orthogonal sum diagram

$$\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$$

of $\mathfrak{HLoc}$ lying in $\mathfrak{M}$ is said to be an *orthogonal* $\mathfrak{M}$-*sum diagram* if for any partial orthogonal sum diagram

$$\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}'_\lambda} \mathbf{X}'\}_{\lambda \in \Lambda}$$

of $\mathfrak{HLoc}$ lying in $\mathfrak{M}$ the unique morphism $\mathbf{g}$ of $\mathfrak{HLoc}$ such that $\mathscr{H}(\mathbf{g})$ is a partial isometry whose initial space is the orthogonal sum of $\mathscr{I}(\mathscr{H}(\mathbf{f}'_\lambda))$'s and $\mathbf{g} \circ \mathbf{f}_\lambda = \mathbf{f}'_\lambda$ for any $\lambda \in \Lambda$ belongs to $\mathfrak{M}$, in which $\mathbf{X}$ is called an orthogonal $\mathfrak{M}$-sum of $\mathbf{X}_\lambda$'s and is denoted by $\sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$. If Λ is a finite set, say, $\Lambda = \{1, 2\}$, then such a notation as $\mathbf{X}_1 \otimes_{\mathfrak{M}} \mathbf{X}_2$ is preferred. Note that an $\mathfrak{M}$-trivial Hilbert locale, if it exists, can be regarded as an orthogonal $\mathfrak{M}$-sum of the empty family of Hilbert locales in $\mathfrak{M}$. An embedding $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$ is called an $\mathfrak{M}$-*embedding* if it can be completed to an orthogonal $\mathfrak{M}$-sum diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Z}$$

A surjection $\mathbf{f}$: $\mathbf{X} \to \mathbf{X}$ in $\mathfrak{M}$ is called an $\mathfrak{M}$-surjection if $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$. Given Hilbert locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, if there exists an $\mathfrak{M}$-embedding $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ (an $\mathfrak{M}$-surjection $\mathbf{f}$: $\mathbf{Y} \to \mathbf{X}$, resp.), then we say that $\mathbf{X}$ is an $\mathfrak{M}$-*sublocale* ($\mathfrak{M}$-*quotient*, resp.) of $\mathbf{Y}$.

A *manual of Hilbert locales* is a small subcategory $\mathfrak{M}$ of the category $\mathfrak{HLoc}$ abiding by the following conditions:

(3.1) For any pair $(\mathbf{X}, \mathbf{Y})$ of Hilbert locales in $\mathfrak{M}$, there exists at most a sole morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$.

(3.2) For any Hilbert locales $\mathbf{X}, \mathbf{Y}$ in $\mathfrak{M}$, if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ for any Hilbert locale $\mathbf{Z}$ in $\mathfrak{M}$.

(3.3) There exists at least an $\mathfrak{M}$-trivial Hilbert locale in $\mathfrak{M}$.

(3.4) For any Hilbert locales $\mathbf{X}, \mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, there exists a Hilbert locale $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$.

(3.5) For any Hilbert locale $\mathbf{Z}$ with $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{W}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{W}$ imply $\mathbf{Z} \perp_{\mathfrak{M}} \mathbf{W}$ for any Hilbert locale $\mathbf{W}$ in $\mathfrak{M}$.

(3.6) For any Hilbert locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ iff there exists a Hilbert locale $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal.

(3.7) For any Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$, if $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{X}$, then $\mathbf{X}$ is $\mathfrak{M}$-trivial.

(3.8) For any commutative diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \\ & {}_{\mathbf{g}}\searrow \quad \nearrow_{\mathbf{h}} & \\ & \mathbf{Z} & \end{array}$$

of $\mathfrak{HLoc}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{g}$ is an $\mathfrak{M}$-surjection, then $\mathbf{h}$ is in $\mathfrak{M}$.

(3.9) For any commutative diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \\ & {}_{\mathbf{g}}\searrow \quad \nearrow_{\mathbf{h}} & \\ & \mathbf{Z} & \end{array}$$

of $\mathfrak{HLoc}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{h}$ is an $\mathfrak{M}$-embedding, then $\mathbf{g}$ is in $\mathfrak{M}$.

Proposition 3.1. Any manual $\mathfrak{M}$ of Hilbert locales abides by the following condition:

(3.10) For any finite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal Hilbert locales in $\mathfrak{M}$, $\sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ exists.

Proof. If the number n of elements in Λ is 0, then any $\mathfrak{M}$-trivial Hilbert locale whose existence is guaranteed by condition (3.3) fills the role of the desired orthogonal $\mathfrak{M}$-sum. If $n = 1$, the condition is trivially satisfied. If $n = 2$, the condition is no other than condition (3.4), which any manual of Hilbert locales should satisfy. If $n \geq 3$, say, if $n = 4$ and $\Lambda = \{1, 2, 3, 4\}$, then by repeated application of conditions (3.4) and (3.5) we have $((\mathbf{X}_1 \oplus_{\mathfrak{M}} \mathbf{X}_2) \oplus_{\mathfrak{M}} \mathbf{X}_3) \oplus_{\mathfrak{M}} \mathbf{X}_4$, which is easily seen to play the role of the desired orthogonal $\mathfrak{M}$-sum of the family. ■

A manual $\mathfrak{M}$ of Hilbert locales is called *σ-coherent* if it satisfies the following condition besides the above ones:

$(3.10)_\sigma$ For any sequence $\{\mathbf{X}_i\}_{i \in \mathbf{N}}$ of pairwise $\mathfrak{M}$-orthogonal Hilbert locales in $\mathfrak{M}$, there exists a Hilbert locale $\mathbf{Z}$ such that $\mathbf{Z} = \sum_{i \in \mathbf{N}} \oplus_{\mathfrak{M}} \mathbf{X}_i$.

A manual $\mathfrak{M}$ of Hilbert locales is said to be *completely coherent* if it satisfies the following condition:

$(3.10)_\infty$ For any infinite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal Hilbert locales in $\mathfrak{M}$, there exists a Hilbert locale $\mathbf{Z}$ in $\mathfrak{M}$ with $\mathbf{Z} = \sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$.

With such a bewilderingly abstract concept as our brand-new one of a manual of Hilbert locales just introduced, we should give the reader a feel for it by examples before subjecting it to theoretical scrutiny in earnest. A Hilbert space $\mathscr{H}$ gives two concomitant instances. Let us begin with the prosaic one.

Example 3.2. Let $\mathscr{H}$ be a Hilbert space. Our first-class manual $\mathfrak{M}_{\mathscr{H}}$ of Hilbert locales on $\mathscr{H}$ has as objects the Hilbert locales $\mathbf{X}$ whose duals $\mathscr{H}(\mathbf{X})$ are closed linear subspaces of $\mathscr{H}$. We decree that a morphism $\mathbf{f}: \mathbf{X}_1 \to \mathbf{X}_2$ of $\mathfrak{HLoc}$ with $\mathscr{H}(\mathbf{X}_1)$ and $\mathscr{H}(\mathbf{X}_2)$ being closed linear subspaces of $\mathscr{H}$ passes for a morphism of $\mathfrak{M}_{\mathscr{H}}$ iff $\mathscr{H}(\mathbf{X}_1) \subseteq \mathscr{H}(\mathbf{X}_2)$ and $\mathscr{H}(\mathbf{f})$ is the orthogonal projection of $\mathscr{H}(\mathbf{X}_2)$ onto $\mathscr{H}(\mathbf{X}_1)$. It is easy to see that Hilbert locales $\mathbf{X}_1$ and $\mathbf{X}_2$ in $\mathfrak{M}_{\mathscr{H}}$ are $\mathfrak{M}_{\mathscr{H}}$-orthogonal iff the closed linear subspaces $\mathscr{H}(\mathbf{X}_1)$ and $\mathscr{H}(\mathbf{X}_2)$ are orthogonal in $\mathscr{H}$. It is also easy to see that Hilbert locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{\mathscr{H}}$ are $\mathfrak{M}_{\mathscr{H}}$-equivalent iff $\mathbf{X} = \mathbf{Y}$. A Hilbert locale $\mathbf{X}$ in $\mathfrak{M}_{\mathscr{H}}$ is $\mathfrak{M}_{\mathscr{H}}$-maximal iff $\mathscr{H}(\mathbf{X}) = \mathscr{H}$. It is easy to see that Hilbert locales $\mathbf{X}_1$ and $\mathbf{X}_2$ are $\mathfrak{M}_{\mathscr{H}}$-equivalent iff $\mathbf{X}_1 = \mathbf{X}_2$.

The same Hilbert space $\mathscr{H}$ can give a far and away more flamboyant example, which one hopes may delight the appreciative reader.

Example 3.3. Let $\mathscr{H}$ be a Hilbert space. Our second-class manual $\mathfrak{M}_{[\mathscr{H}]}$ of Hilbert locales on $\mathscr{H}$ has as objects the Hilbert locales $\mathbf{X}$ whose duals $\mathscr{H}(\mathbf{X})$ are of the form $\mathscr{H}_1/\mathscr{K}_1$, where $\mathscr{H}_1$ and $\mathscr{K}_1$ are closed linear subspaces of $\mathscr{H}$ with $\mathscr{K}_1 \subseteq \mathscr{H}_1$, and $\mathscr{H}_1/\mathscr{K}_1$ denotes the quotient Hilbert space. Note that $\mathscr{H}_1/\mathscr{K}_1$ and the orthogonal complement $\mathscr{H}_1 \ominus \mathscr{K}_1$ of $\mathscr{K}_1$ in $\mathscr{H}_1$ are canonically isomorphic in $\mathfrak{Hil}$. We decree that a morphism $\mathbf{f}: \mathbf{X}_1 \to \mathbf{X}_2$ of $\mathfrak{HLoc}$ with $\mathscr{H}(\mathbf{X}_1) = \mathscr{H}_1/\mathscr{K}_1$ and $\mathscr{H}(\mathbf{X}_2) = \mathscr{H}_2/\mathscr{K}_2$ passes for a morphism of $\mathfrak{M}_{[\mathscr{H}]}$ iff $\mathscr{H}_1 \subseteq \mathscr{H}_2$, $\mathscr{K}_1 \subseteq \mathscr{K}_2$, and $\mathscr{H}(\mathbf{f})$ is the restriction to $\mathscr{H}_2 \ominus \mathscr{K}_2$ of the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_1 \ominus \mathscr{K}_1$, where $\mathscr{H}_1/\mathscr{K}_1$ ($\mathscr{H}_2/\mathscr{K}_2$, resp.) and $\mathscr{H}_1 \ominus \mathscr{K}_1$ ($\mathscr{H}_2 \ominus \mathscr{K}_2$, resp.) are canonically identified. Note that in this case, since $\mathscr{K}_1$ and $\mathscr{H}_2 \ominus \mathscr{K}_2$ are orthogonal. $\mathscr{H}(\mathbf{f})$ is also the restriction to $\mathscr{H}_2 \ominus \mathscr{K}_2$ of the orthogonal projection of $\mathscr{H}$ onto $\mathscr{K}_1$. Thus it is easy to see that the morphisms of $\mathfrak{M}_{[\mathscr{H}]}$ are indeed closed under composition. Note that a morphism $\mathbf{f}: \mathbf{X}_1 \to \mathbf{X}_2$ of $\mathfrak{M}_{[\mathscr{H}]}$ with $\mathscr{H}(\mathbf{X}_1) = \mathscr{H}_1/\mathscr{K}_1$ and $\mathscr{H}(\mathbf{X}_2) = \mathscr{H}_2/\mathscr{K}_2$ is an embedding iff $\mathscr{H}_1 \ominus \mathscr{K}_1 \subseteq \mathscr{H}_2 \ominus \mathscr{K}_2$ and $\mathscr{H}(\mathbf{f})$ is the orthogonal projection of $\mathscr{H}_2 \ominus \mathscr{K}_2$ onto $\mathscr{H}_1 \ominus \mathscr{K}_1$. Note also that the morphism $\mathbf{f}$ is a surjection iff $\mathscr{H}_2 \ominus \mathscr{K}_2 \subseteq \mathscr{H}_1 \ominus \mathscr{K}_1$ and $\mathscr{H}(\mathbf{f})$ is the identity on $\mathscr{H}_2 \ominus \mathscr{K}_2$. It is easy to see that Hilbert locales $\mathbf{X}_1$ and $\mathbf{X}_2$ in $\mathfrak{M}_{[\mathscr{H}]}$ with $\mathscr{H}(\mathbf{X}_1) = \mathscr{H}_1/\mathscr{K}_1$ and $\mathscr{H}(\mathbf{X}_2) = \mathscr{H}_2/\mathscr{K}_2$ are $\mathfrak{M}_{[\mathscr{H}]}$-orthogonal iff $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal, in which

$$\mathscr{H}(\mathbf{X}_1 \oplus_{\mathfrak{M}_{[\mathscr{H}]}} \mathbf{X}_2) = (\mathscr{H}_1 \oplus \mathscr{H}_2)/(\mathscr{K}_1 \oplus \mathscr{K}_2)$$

It is also easy to see that they are $\mathfrak{M}_{[\mathscr{H}]}$-equivalent iff $\mathscr{H}_1 = \mathscr{H}_2$. A Hilbert locale $\mathbf{X}_1$ or $\mathfrak{M}_{[\mathscr{H}]}$ with $\mathscr{H}(\mathbf{X}_1) = \mathscr{H}_1/\mathscr{K}_1$ is $\mathfrak{M}_{[\mathscr{H}]}$-maximal iff $\mathscr{H}_1 - \mathscr{H}$.

Now we give a method for constructing a new manual of Hilbert locales from given ones, which will be used in the succeeding section.

Example 3.4. Let $\{\mathfrak{M}_\lambda\}_{\lambda \in \Lambda}$ be a family of manuals of Hilbert locales. The orthogonal sum $\bigoplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda$ of $\mathfrak{M}_\lambda$'s is a subcategory of $\mathfrak{HLoc}$ whose objects are the dual objects $\bigoplus_{\lambda \in \Lambda} \mathbf{X}_2$ of $\bigoplus_{\lambda \in \Lambda} \mathscr{H}(\mathbf{X}_\lambda)$ with $\mathbf{X}_\lambda$ a Hilbert locale in $\mathfrak{M}_\lambda$ for each $\lambda \in \Lambda$ and whose morphisms are the dual morphisms $\bigoplus_{\lambda \in \Lambda} \mathbf{f}_\lambda$ of $\bigoplus_{\lambda \in \Lambda} \mathscr{H}(\mathbf{f}_\lambda)$ with $\mathbf{f}_\lambda$ a morphism in $\mathfrak{M}_\lambda$ for each $\lambda \in \Lambda$. It is easy to see that $\bigoplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda$ is indeed a manual of Hilbert locales. Note that $\bigoplus_{\lambda \in \Lambda} \mathbf{X}_\lambda$ and $\bigoplus_{\lambda \in \Lambda} \mathbf{Y}_\lambda$ are $(\bigoplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda)$-orthogonal iff $\mathbf{X}_\lambda$ and $\mathbf{Y}_\lambda$ are $\mathfrak{M}_\lambda$-orthogonal for all $\lambda \in \Lambda$, in which

$$(\bigoplus_{\lambda \in \Lambda} \mathbf{X}_\lambda) \oplus_{\oplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda} (\bigoplus_{\lambda \in \Lambda} \mathbf{Y}_\lambda) = \bigoplus_{\lambda \in \Lambda} (X_\lambda \oplus_{\mathfrak{M}_\lambda} Y_\lambda)$$

Note also that $\bigoplus_{\lambda \in \Lambda} \mathbf{X}_\lambda$ is $(\bigoplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda)$-maximal iff $\mathbf{X}_\lambda$ is $\mathfrak{M}_\lambda$-maximal for all $\lambda \in \Lambda$. It is easy to see that $\bigoplus_{\lambda \in \Lambda} \mathbf{X}_\lambda$ and $\bigoplus_{\lambda \in \Lambda} \mathbf{Y}_\lambda$ are $(\bigoplus_{\lambda \in \Lambda} \mathfrak{M}_\lambda)$-equivalent iff $\mathbf{X}_\lambda$ and $\mathbf{Y}_\lambda$ are $\mathfrak{M}_\lambda$-equivalent for all $\lambda \in \Lambda$.

Now we are going to show that a manual $\mathfrak{M}$ of Hilbert locales, which shall be fixed for a while, gives rise to an orthocoherent associative orthoalgebra $\mathscr{L}(\mathfrak{M}) = (L_{\mathfrak{M}}, +_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$, for which several ancillary results are in order.

Proposition 3.5. All the $\mathfrak{M}$-trivial Hilbert locales are mutually $\mathfrak{M}$-equivalent.

Proof. Let $\mathbf{X}, \mathbf{Y}$ be $\mathfrak{M}$-trivial Hilbert locales. Let $\mathbf{Z}$ be an arbitrary Hilbert locale in $\mathfrak{M}$. Since $\mathbf{X}$ is $\mathfrak{M}$-trivial, there exists a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$ so that by condition (3.2), if $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, then $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$. By changing the roles of $\mathbf{X}$ and $\mathbf{Y}$, are also certain that if $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Hence $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$. ■

We denote by $0_{\mathfrak{M}}$ their $\mathfrak{M}$-equivalence class.

Proposition 3.6. Any $\mathfrak{M}$-trivial Hilbert locale $\mathbf{X}$ is $\mathfrak{M}$-orthogonal to any Hilbert locale $\mathbf{Y}$ in $\mathfrak{M}$.

Proof. Since $\mathbf{X}$ is $\mathfrak{M}$-trivial, there exists a unique morphism $\mathbf{f}: \mathbf{X} \to \mathbf{X}$ in $\mathfrak{M}$. Thus the orthogonal sum diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{1_{\mathbf{Y}}} \mathbf{Y}$$

lies in $\mathfrak{M}$, which shows that $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$. ■

Proposition 3.7. There exists an $\mathfrak{M}$-maximal Hilbert locale in $\mathfrak{M}$.

Proof. By condition (3.3) there exists at least an $\mathfrak{M}$-trivial Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$. Since obviously $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{X}$, the desired conclusion follows from condition (3.6). ■

Proposition 3.8. Every Hilbert locale $\mathbf{X}$ in $0_{\mathfrak{M}}$ is $\mathfrak{M}$-trivial.

Proof. By Proposition 3.7 there exists an $\mathfrak{M}$-maximal Hilbert locale $\mathbf{Y}$ in $\mathfrak{M}$. By Proposition 3.6 we have $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, which implies that $\mathbf{X}$ is $\mathfrak{M}$-trivial. ■

Proposition 3.9. All the $\mathfrak{M}$-maximal Hilbert locales in $\mathfrak{M}$ are $\mathfrak{M}$-equivalent. Every Hilbert locale of $\mathfrak{M}$ which is $\mathfrak{M}$-equivalent to an $\mathfrak{M}$-maximal Hilbert locale is also $\mathfrak{M}$-maximal.

Proof. By Proposition 3.6 the $\mathfrak{M}$-maximal Hilbert locales of $\mathfrak{M}$ can be characterized as the Hilbert locales of $\mathfrak{M}$ to which exactly the $\mathfrak{M}$-trivial Hilbert locales are $\mathfrak{M}$-orthogonal. ■

We denote by $1_{\mathfrak{M}}$ the class of all the $\mathfrak{M}$-maximal Hilbert locales in $\mathfrak{M}$.

Proposition 3.10. For any Hilbert locales $\mathbf{X}, \mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, we have that for any Hilbert locale $\mathbf{Z}$, $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$.

Proof. This follows forthwith from conditions (3.2) and (3.5). ■

Corollary 3.11. Whenever $\mathbf{X}, \mathbf{X}', \mathbf{Y}$, and $\mathbf{Y}'$ are Hilbert locales in $\mathfrak{M}$ such that $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{X}'$ and $\mathbf{Y} \simeq_{\mathfrak{M}} \mathbf{Y}'$, then $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ iff $\mathbf{X}' \perp_{\mathfrak{M}} \mathbf{Y}'$, in which any orthogonal $\mathfrak{M}$-sum of $\mathbf{X}$ and $\mathbf{Y}$ is $\mathfrak{M}$-equivalent to any orthogonal $\mathfrak{M}$-sum of $\mathbf{X}'$ and $\mathbf{Y}'$.

Let $L_{\mathfrak{M}} = \{[\mathbf{X}]_{\mathfrak{M}} | \mathbf{X}$ is a Hilbert locale in $\mathfrak{M}\}$. By the above corollary we can safely decree that $[\mathbf{X}]_{\mathfrak{M}} +_{\mathfrak{M}} [\mathbf{Y}]_{\mathfrak{M}}$ is defined iff $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, in which $[\mathbf{X}]_{\mathfrak{M}} +_{\mathfrak{M}} [\mathbf{Y}]_{\mathfrak{M}}$ is defined to be $[\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}]_{\mathfrak{M}}$. Now we have to show the following.

Theorem 3.12. The structure $\mathscr{L}(\mathfrak{M}) = (L_{\mathfrak{M}}, +_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$ thus defined is indeed an orthocoherent associative orthoalgebra.

Proof. It suffices to note the following:

(a) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, then $\mathbf{X} +_{\mathfrak{M}} \mathbf{Y} \simeq_{\mathfrak{M}} \mathbf{Y} +_{\mathfrak{M}} \mathbf{X}$.

(b) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ and $\mathbf{X} +_{\mathfrak{M}} \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y} +_{\mathfrak{M}} \mathbf{Z}$, and

$$(\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}) \oplus_{\mathfrak{M}} \mathbf{Z} \simeq_{\mathfrak{M}} \mathbf{X} \oplus_{\mathfrak{M}} (\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z})$$

(c) For any $\mathbf{X}$ in $\mathfrak{M}$, there exists $\mathbf{Y}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ and $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ is $\mathfrak{M}$-maximal. If $\mathbf{Y}'$ satisfies the same property, then $\mathbf{Y} \simeq_{\mathfrak{M}} \mathbf{Y}'$.

(d) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{X}$, then $\mathbf{X}$ is $\mathfrak{M}$-trivial.

(e) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}, \mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, then $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$.

Statements (a), (b), and (e) are obvious by Proposition 3.10. Statement (c) follows from condition (3.6), while statement (d) follows from condition (3.7). ■

The associative orthoalgebra $\mathscr{L}(\mathfrak{M})$ thus obtained is called the *associative orthoalgebra associated with* $\mathfrak{M}$. It is well known that the notions of an orthocoherent orthoalgebra and an orthomodular poset are essentially equivalent concepts, for which the reader is referred to Gudder (1988, Corollary 3.4 and Theorem 3.5). The orthomodular poset $\mathscr{Q}(\mathfrak{M}) = (L_{\mathfrak{M}}, \leq_{\mathfrak{M}}, \urcorner_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$ corresponding to $\mathscr{L}(\mathfrak{M})$ is called the *orthomodular poset associated with* $\mathfrak{M}$. It is easy to see the following.

Proposition 3.13. If a manual $\mathfrak{M}$ of Hilbert locales is σ-coherent (completely coherent, resp.), then the orthomodular poset $\mathscr{Q}(\mathfrak{M})$ associated with $\mathfrak{M}$ is σ-orthocomplete (orthocomplete, resp.).

The following proposition is also of some interest.

Proposition 3.14. For any isomorphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{H}\mathfrak{Loc}$ lying in $\mathfrak{M}$, its inverse $\mathbf{f}^{-1}$ belongs to $\mathfrak{M}$ iff $\mathbf{f}$ is an $\mathfrak{M}$-embedding.

Proof. To see the if part of the above statement, consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Y} & \xrightarrow{1_{\mathbf{Y}}} & \mathbf{Y} \\ & {\scriptstyle \mathbf{f}^{-1}}\searrow \quad \nearrow{\scriptstyle \mathbf{f}} & \\ & \mathbf{X} & \end{array}$$

Then the desired conclusion follows from condition (3.9). To see the only-if part of the statement, let $\mathbf{W}$ be an $\mathfrak{M}$-trivial Hilbert locale in $\mathfrak{M}$. Then the diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \leftarrow \mathbf{W}$$

in $\mathfrak{M}$ is an orthogonal $\mathfrak{M}$-sum diagram, since for any morphism $\mathbf{g}: \mathbf{X} \to \mathbf{Z}$ of $\mathfrak{M}$ with $\mathscr{H}(\mathbf{g})$ a partial isometry, we have the following commutative diagram in $\mathfrak{M}$:

$$\begin{array}{ccccc} & & \mathbf{Z} & & \\ & {\scriptstyle \mathbf{g}}\nearrow & \uparrow{\scriptstyle \mathbf{g}\circ\mathbf{f}^{-1}} & \nwarrow & \\ \mathbf{X} & \xrightarrow[\mathbf{f}]{} & \mathbf{Y} & \longleftarrow & \mathbf{W} \end{array}$$

By the same token we have the following dual result.

Proposition 3.15. For any isomorphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{H}\mathfrak{Loc}$ lying in $\mathfrak{M}$, its inverse $\mathbf{f}^{-1}$ belongs to $\mathfrak{M}$ iff $\mathbf{f}$ is an $\mathfrak{M}$-surjection.

Now we are in a position to discuss morphisms. In the next section we need to discuss representations of manuals of Boolean locales over manuals of Hilbert locales, so that whether each of $\mathfrak{M}$ and $\mathfrak{N}$ is a manual of Boolean locales or a manual of Hilbert locales, we ought to define the notion of a *morphism* $F: \mathfrak{M} \to \mathfrak{N}$, which is a functor satisfying the following conditions:

(3.11) If $\mathbf{X}$ is $\mathfrak{M}$-trivial, then $F(\mathbf{X})$ is $\mathfrak{N}$-trivial.

(3.12) If $\mathbf{X}$ is $\mathfrak{M}$-maximal, then $F(X)$ is $\mathfrak{N}$-maximal.

(3.13) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, then $F(\mathbf{X}) \perp_{\mathfrak{N}} F(\mathbf{Y})$ and $F(\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}) = F(\mathbf{X}) \oplus_{\mathfrak{N}} F(\mathbf{Y})$.

In the above and henceforth, if $\mathfrak{M}$ is a manual of Boolean locales, then $\mathfrak{M}$-triviality and triviality are used interchangeably, though they are apparently distinct concepts in the case that $\mathfrak{M}$ is a manual of Hilbert locales.

The morphism F is called *σ-orthocomplete* (*orthocomplete*, resp.) if it satisfies the following condition $(3.13)_\sigma$ [$(3.13)_\infty$, resp.]:

$(3.13)_\sigma$ If $\mathbf{Y} = \sum_{i \in \mathrm{N}} \oplus_{\mathfrak{M}} \mathbf{X}_i$ with $\{\mathbf{X}_i\}_{i \in \mathrm{N}}$ a sequence of pairwise $\mathfrak{M}$-orthogonal Boolean or Hilbert locales in $\mathfrak{M}$, then $F(\mathbf{Y}) = \sum_{i \in \mathrm{N}} \oplus_{\mathfrak{N}} F(\mathbf{X}_i)$.

$(3.13)_\infty$ If $\mathbf{Y} = \sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ with $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ an infinite family of pairwise $\mathfrak{M}$-orthogonal Boolean or Hilbert locales in $\mathfrak{M}$, then $F(\mathbf{Y}) = \sum_{\lambda \in \Lambda} \oplus_{\mathfrak{N}} F(\mathbf{X}_\lambda)$.

A morphism $F: \mathfrak{M} \to \mathfrak{N}$ of manuals is said to be *faithful* if for any objects $\mathbf{X}, \mathbf{Y}$ in $\mathfrak{M}$, $F(\mathbf{X}) \perp_{\mathfrak{N}} F(\mathbf{Y})$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$.

Proposition 3.16. If each of $\mathfrak{M}$ and $\mathfrak{N}$ is a manual of Boolean or Hilbert locales, and $F: \mathfrak{M} \to \mathfrak{N}$ is a morphism of manuals, then $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$ always implies $F(\mathbf{X}) \simeq_{\mathfrak{N}} F(\mathbf{Y})$ in $\mathfrak{N}$.

Proof. Since $X \simeq_{\mathfrak{M}} \mathbf{Y}$ by assumption, we have by condition (3.6) of this paper or by condition (3.9) of Nishimura (1993*b*) and object $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}, \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal. This means by conditions (3.12) and (3.13) that $F(\mathbf{X}) \perp_{\mathfrak{N}} F(\mathbf{Z})$, $F(\mathbf{Y}) \perp_{\mathfrak{N}} F(\mathbf{Z})$, and both of $F(\mathbf{X}) \oplus_{\mathfrak{N}} F(\mathbf{Z})$ and $F(\mathbf{Y}) \otimes_{\mathfrak{N}} F(\mathbf{Z})$ are $\mathfrak{N}$-maximal, which implies by condition (3.6) of this paper or by condition (3.9) of Nishimura (1993*b*) again that $F(\mathbf{X}) \simeq_{\mathfrak{N}} F(\mathbf{Y})$. ■

By this proposition we can see easily that a morphism of manuals naturally induces a homomorphism of their associated associative orthoalgebras and a morphism of their associated orthomodular posets. In particular, if two manuals are isomorphic, their associated associative orthoalgebras as well as their associated orthomodular posets are isomorphic.

In case both of $\mathfrak{M}$ and $\mathfrak{N}$ are manuals of Hilbert locales, we need a stronger notion of a morphism in the next section. An *isometric morphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a pair (F, τ), where $F: \mathfrak{M} \to \mathfrak{N}$ is a morphism of manuals of Hilbert locales and τ is an assignment to each object $\mathbf{X}$ in $\mathfrak{M}$ of a morphism $\tau_{\mathbf{X}}: F(\mathbf{X}) \to \mathbf{X}$ of $\mathfrak{H}\mathfrak{Loc}$ abiding by the following conditions.

(3.14) $\tau_{\mathbf{X}}$ is a surjection in $\mathfrak{H}\mathfrak{Loc}$ for each object $\mathbf{X}$ in $\mathfrak{M}$.

(3.15) τ is natural in the sense that for any morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$, the diagram

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{\tau_{\mathbf{X}}} & F(\mathbf{X}) \\ {\scriptstyle \mathbf{f}}\downarrow & & \uparrow{\scriptstyle f(\mathbf{f})} \\ \mathbf{Y} & \xleftarrow{\tau_{\mathbf{Y}}} & F(\mathbf{Y}) \end{array}$$

is commutative.

The notion of a *state* on a manual $\mathfrak{M}$ of Hilbert locales is defined in the same way as that on a manual of Boolean locales was. Given a morphism $\mathbf{F}: \mathfrak{M} \to \mathfrak{N}$ of manuals, we say that a state ω on $\mathfrak{M}$ and a state ω' are F-related if for any object $\mathbf{X}$ in $\mathfrak{M}$, $\omega'(F(\mathbf{X})) = \omega(\mathbf{X})$. A manual $\mathfrak{M}$ of Boolean or Hilbert locales is said to *admit* a *full set of states* if for any object $\mathbf{X}$ in $\mathfrak{M}$ which is not $\mathfrak{M}$-trivial, there exists a state ω such that $\omega(\mathbf{X}) \neq 0$.

Given a manual $\mathfrak{M}$ of Hilbert locales, a *vector field* on $\mathfrak{M}$ is an assignment $\mathbb{x}$ to each Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$ of a vector $\mathbb{x}_{\mathbf{X}}$ in $\mathscr{H}(\mathbf{X})$ such that for any orthogonal $\mathfrak{M}$-sum diagram

$$\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{X}\}_{\lambda \in \Lambda}$$

in $\mathfrak{M}$, $\mathbb{x}_{\mathbf{X}_\lambda} = \mathscr{H}(\mathbf{f}_\lambda)(\mathbb{x}_{\mathbf{X}})$. A vector field $\mathbb{x}$ on $\mathfrak{M}$ is called *normalized* if $\|\mathbb{x}_{\mathbf{X}}\| = 1$ for any $\mathfrak{M}$-maximal Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$. It is easy to see that given a normalized vector field $\mathbb{x}$ on $\mathfrak{M}$, the assignment to each Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$ of $\|\mathbb{x}_{\mathbf{X}}\|^2$ is a state, which is denoted by $\omega_{\mathbb{x}}$. Given an isometric orthocomplete morphism $(F, \tau): \mathfrak{M} \to \mathfrak{N}$ of manuals of Hilbert locales, a vector field $\mathbb{x}$ on $\mathfrak{M}$ and a vector field $\mathbb{y}$ on $\mathfrak{N}$ are said to be *(F, τ)-related* provided that for any Hilbert locale $\mathbf{X}$ in $\mathfrak{M}$, $\mathscr{H}(\tau_{\mathbf{X}})(\mathbb{x}_{\mathbf{X}}) = \mathbb{y}_{F(\mathbf{X})}$. In this case, if $\mathbb{x}$ and $\mathbb{y}$ are normalized as well, then $\omega_{\mathbb{x}}$ and $\omega_{\mathbb{y}}$ are F-related states.

4. THE GNS THEOREM

Let $\mathfrak{M}$ be a manual of Boolean locales in the strong sense, which shall be fixed throughout this section. Let ω be a state on $\mathfrak{M}$. By Corollary 2.6 the state ω induces its corresponding state on the orthomodular poset $\mathscr{Q}(\mathfrak{M})$ associated with $\mathfrak{M}$, which we denote by $\bar{\omega}$. As we have seen in Section 4 of Nishimura (1993*b*), the dual $\mathscr{P}(\mathbf{X})$ of each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ is canonically isomorphic with a relative complete Boolean subalgebra $\mathbf{B}_{\mathbf{X}}$ of $\mathscr{Q}(\mathfrak{M})$. In particular, the Stonean spaces of $\mathscr{P}(\mathbf{X})$ and $\mathbf{B}_{\mathbf{X}}$ can be identified and are denoted by $\Xi_{\mathbf{X}}$. Thus the restriction of $\bar{\omega}$ to $\mathbf{B}_{\mathbf{X}}$ naturally induces a Borel measure $\omega_{\mathbf{X}}$ on the Stonean space $\Xi_{\mathbf{X}}$ of $\mathscr{P}(\mathbf{X})$. We denote by $L^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ the Hilbert space of square-integrable complex-valued functions on $\Xi_{\mathbf{X}}$. By what is admittedly called Stone duality between Boolean algebras and Boolean spaces, the opposite $\mathscr{P}(\mathbf{f})$ of each morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$ induces a continuous function $\Xi_{\mathbf{f}}: \Xi_{\mathbf{X}} \to \Xi_{\mathbf{Y}}$. It is not difficult to see that the assignment to each $f \in L^2(\Xi_{\mathbf{Y}}, \omega_{\mathbf{Y}})$ of $f \circ \Xi_{\mathbf{f}}$ gives rise to a contractive linear mapping $T_{\mathbf{f}}$ from $L^2(\Xi_{\mathbf{Y}}, \omega_{\mathbf{Y}})$ to $L^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$. The totality of $L^2(\Xi_X, \omega_{\mathbf{X}})$'s and $T_{\mathbf{f}}$'s for all Hilbert locales $\mathbf{X}$ and all morphisms $\mathbf{f}$ in $\mathfrak{M}$ gives a subcategory of $\mathfrak{Hil}$, whose dual category is denoted by $\mathfrak{H}(\mathfrak{M}, \omega)$. It is easy to see the following.

Proposition 4.1. $\mathfrak{H}(\mathfrak{M}, \omega)$ is a completely coherent manual of Hilbert locales.

We denote by $\mathbb{x}(\mathfrak{M}, \omega)$ the assignment to each object $L^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ in $\mathfrak{H}(\mathfrak{M}, \omega)^{\mathrm{op}}$ of the constant function 1 on $\Xi_{\mathbf{X}}$. Then it is easy to see the following.

Proposition 4.2. $\mathbb{x}(\mathfrak{M}, \omega)$ is a normalized vector field on $\mathfrak{H}(\mathfrak{M}, \omega)$, so that it induces a state $\omega_{\mathbb{x}(\mathfrak{M},\omega)}$ on $\mathfrak{H}(\mathfrak{M}, \omega)$.

We denote by $F^{\omega}_{\mathfrak{M}}$ the functor on $\mathfrak{M}$ such that $F^{\omega}_{\mathfrak{M}}(\mathbf{X})^{\mathrm{op}} = L^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and $F^{\omega}_{\mathfrak{M}}(\mathbf{f})^{\mathrm{op}} = T_{\mathbf{f}}$ for each morphism $\mathbf{f}$ in $\mathfrak{M}$. It is easy to see the following:

Proposition 4.3. $F^{\omega}_{\mathfrak{M}}$ is an orthocomplete morphism from the manual $\mathfrak{M}$ of Boolean locales to the manual $\mathfrak{H}(\mathfrak{M}, \omega)$ of Hilbert locales, in which the states ω and $\omega_{\mathbb{x}(\mathfrak{M},\omega)}$ are $F^{\omega}_{\mathfrak{M}}$-related.

Now we are ready to present our GNS theorem.

Theorem 4.4. For any manual $\mathfrak{N}$ of Hilbert locales with a vector field $\mathbb{y}$ on it and any orthocomplete morphism G from $\mathfrak{M}$ to $\mathfrak{N}$ in which ω and $\omega_{\mathbb{y}}$ are G-related, there exists a unique isometric orthocomplete morphism (H, τ) from $\mathfrak{H}(\mathfrak{M}, \omega)$ to $\mathfrak{N}$ abiding by the following conditions:

(4.1) Vector fields $\mathbb{x}(\mathfrak{M}, \omega)$ and $\mathbb{y}$ are (H, τ)-related.
(4.2) The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{F^{\omega}_{\mathfrak{M}}} & \mathfrak{H}(\mathfrak{M}, \omega) \\ & {}_{G}\searrow & \downarrow H \\ & & \mathfrak{N} \end{array}$$

Outline of the Proof. The construction of the desired H is all but trivial, so that the major part of the proof consists in the construction of the desired τ. Note that for each Boolean locate $\mathbf{X}$ in $\mathfrak{M}$, the space $SL^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ consisting of all simple functions in $L^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ is a dense linear subspace of $L^2(\Xi_X, \omega_{\mathbf{X}})$. Recall that a function is called *simple* if it takes only a finite number of values. If $\mathbf{X} = \mathbf{X}_1 \oplus_{\mathfrak{M}} \cdots \oplus_{\mathfrak{M}} \mathbf{X}_n$, then $G(\mathbf{X}) = G(\mathbf{X}_1) \oplus_{\mathfrak{N}} \cdots \oplus_{\mathfrak{N}} G(\mathbf{X}_n)$. Let $\alpha_1, \ldots, \alpha_n$ be complex numbers. If desired τ exists, then the putative $\mathscr{H}(\tau_{\mathbf{X}})$ should map a simple function

$$\alpha_1 \chi_{\Xi_{\mathbf{X}_1}} + \cdots + \alpha_n \chi_{\Xi_{\mathbf{X}_n}} \quad \text{on } \Xi_{\mathbf{X}}$$

to the vector

$$\alpha_1 P_{\mathscr{H}(G(\mathbf{X}_1))} \mathbb{y}_{G(\mathbf{X})} + \cdots + \alpha_n P_{\mathscr{H}(G(\mathbf{X}_n))} \mathbb{y}_{G(\mathbf{X})} \qquad \text{of} \quad \mathscr{H}(G(\mathbf{X}))$$

where $\Xi_{\mathbf{X}}$ can canonically be identified with the topological sum of $\Xi_{\mathbf{X}_i}$'s,

$\chi_{\Xi_{\mathbf{X}_i}}$ denotes the characteristic function of $\Xi_{\mathbf{X}_i}$ for each i, $\mathscr{H}(G(\mathbf{X}_i))$ can canonically be identified with the orthogonal sum of $\mathscr{H}(G(\mathbf{X}))$'s, and $P_{\mathscr{H}(G(\mathbf{X}_i))}$ denotes the orthogonal projection of $\mathscr{H}(G(\mathbf{X}))$ onto $\mathscr{H}(G(\mathbf{X}_i))$ for each i. It is not difficult to see that this indeed defines an isometric linear mapping of $SL^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ into $\mathscr{H}(G(\mathbf{X}))$, whose unique isometric linear extension to $L^2(\Xi_{\mathbf{X}}, \omega_{\mathbf{X}})$ should be taken as $\mathscr{H}(\tau_{\mathbf{X}})$. ■

The long and the short of the above theorem is that our construct $\mathfrak{H}(\mathfrak{M}, \omega)$ is universal in some reasonable sense of category theory (MacLane, 1971, Chapter III, §1) and that our vector field $\mathbb{x}(\mathfrak{M}, \omega)$ assumes the role of a cyclic vector in the conventional GNS construction.

The orthogonal-sum construction in Example 3.4 gives the following embedding theorem.

Theorem 4.5. If $\mathfrak{M}$ admits a full set of states, then $\mathfrak{M}$ admits as well a faithful orthocomplete morphism into a completely coherent manual of Hilbert locales.

Outline of the Proof. Take a family $\{\omega_\lambda\}_{\lambda\in\Lambda}$ of states on $\mathfrak{M}$ such that for any nontrivial Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ there exists $\lambda\in\Lambda$ such that $\omega_\lambda(\mathbf{X}) \neq 0$. Take the orthogonal sum $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}(\mathfrak{M}, \omega_\lambda)$ as the desired manual of Hilbert locales. It is not hard to see that the functor

$$\bigoplus_{\lambda\in\Lambda} F_{\mathfrak{M}}^{\omega_\lambda}\colon \quad \mathfrak{M} \to \bigoplus_{\lambda\in\Lambda} \mathfrak{H}(\mathfrak{M}, \omega_\lambda)$$

with

$$\Big(\bigoplus_{\lambda\in\Lambda} F_{\mathfrak{M}}^{\omega_\lambda}\Big)(\mathbf{X}) = \bigoplus_{\lambda\in\Lambda} F_{\mathfrak{M}}^{\omega_\lambda}(\mathbf{X})$$

for each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and

$$\Big(\bigoplus_{\lambda\in\Lambda} F_{\mathfrak{M}}^{\omega_\lambda}\Big)(\mathbf{f}) = \bigoplus_{\lambda\in\Lambda} F_{\mathfrak{M}}^{\omega_\lambda}(\mathbf{f})$$

for each morphism $\mathbf{f}$ in $\mathfrak{M}$ is indeed a desired morphism of manuals. ■

NOTES ADDED IN PROOF

1. Example 3.3 turned out to be incorrect.

2. In Section 4, to make Proposition 4.1 valid, it seems that we should assume that the state ω is centrally supported. The state ω is said to be *centrally supported* if the corresponding state $\bar{\omega}$ on $\mathscr{Q}(\mathfrak{M})$ has a support and it belongs to the center of $\mathscr{Q}(\mathfrak{M})$.

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